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Allocation rules on networks

Rahmi İlkılıç · Çağatay Kayı

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Abstract When allocating a resource, geographical and infrastructural constraints have to be taken into account. We study the problem of distributing a resource through a network from sources endowed with the resource to citizens with claims. A link between a source and a citizen depicts the possibility of a transfer from the source to the citizen. Given the endowments at each source, the claims of citizens, and the network, the question is how to allocate the available resources among the citizens. We consider a simple allocation problem that is free of network constraints, where the total amount can be freely distributed. The simple allocation problem is a *claims problem* where the total amount of claims is greater than what is available. We focus on *resource monotonic* and *anonymous* bilateral principles satisfying a regularity condition and extend these principles to allocation rules on networks. We require the extension to preserve the essence of the bilateral principle for each pair of citizens in the network. We call this condition *pairwise robustness* with respect to the bilateral principle. We provide an algorithm and show that each bilateral principle has a unique extension which is *pairwise robust* (Theorem 1). Next, we consider a Rawlsian criteria of distributive justice and show that there is a unique *"Rawls fair"* rule that equals the extension given by the algorithm (Theorem 2). Pairwise robustness and Rawlsian fairness are two sides of the same coin, the former being a pairwise and the latter a global requirement on the allocation given by a rule. We also show as a corollary that any parametric principle can be extended to an allocation rule (Corollary 1). Finally, we give applications of the algorithm for the egalitarian, the proportional, and the contested garment bilateral principles (Example 1).

R. İlkılıç (✉)
Department of Economics, Bilkent University, Ankara, Turkey
e-mail: rahmi.ilkilic@bilkent.edu.tr

Ç. Kayı
Facultad de Economía, Universidad del Rosario, Bogotá, Colombia

1 Introduction

The world economy has become a densely connected network of supply centers and demand nodes. This is true in particular for natural resources. Given geographical or infrastructural constraints, it is important to understand how scarce resources should be allocated. An example where such network constraints are essential is fresh water resources. As a result of increasing population and developing economies, the need for water is growing immensely. The principal issue is to have an efficient and fair allocation of resources (Ansink and Weikard 2009; Hoekstra 2006). Some other examples are aid relief during natural disasters (Özdamar et al. 2004), common property fisheries (İlkılıç 2011), and the distribution of utilities like electricity and natural gas when there is a supply shock.

We study the problem of distributing a resource through a bipartite network between *citizens* with positive claims (needs or entitlements) and *sources* that are endowed with an amount of the desired resource.[1] If there is a link between a source and a citizen, then the citizen can receive the resource from the source. Each source has a limited endowment of the resource and each citizen has a claim on the resource. Given the network constraints, the claims of citizens, and the endowments at each source, the question is how to allocate the resource among the citizens. An allocation rule assigns to each citizen a quantity of resource satisfying the following feasibility constraints: a citizen can not receive more than his claim and a source can not deliver more than its endowment.

We study those problems where total claims exceeds total endowment.[2] When individuals have claims on a resource that sum up to more than what is available, how should the resource be divided? This problem is a claims problem, formally introduced by O'Neill (1982). Several solutions are commonly used in practice and analyzed in theoretical work (Thomson 2003, 2006).

An *allocation problem* is defined by the endowments of the sources, the claims of the citizens, and the network. First, we define a *simple allocation problem* that is free of network constraints, where the total amount can be freely distributed between the agents. The simple allocation problem is in fact a claims problem. A *principle* is a standard of judgement[3]. which assigns a division of the total endowment between the citizens for any simple allocation problem. We focus on bilateral principles on two-citizen simple allocation problems. A bilateral principle is *resource monotonic* if when the total endowment increases, each citizen receives at least as much as she did initially. A bilateral principle is *anonymous* if the allocation given by the bilateral principle does not depend on the names of the citizens.

The next property is for principles. Suppose a principle has been applied to a simple problem and some citizens leave with what they are prescribed by the principle and the total amount prescribed to these citizens by the principle reduced from the endowment.

[1] For example, when the resource in question is fresh water, the sources are lakes, rivers, dams, etc. and the citizens are cities.

[2] If there is a group of agents on the network whose claims can be completely satisfied without any burden on others, we can simply take those agents out of the network and focus on the "genuine" problem.

[3] For example, the egalitarian principle, the proportional principle, the equal losses principle, etc.

If we apply the principle to the problem with the remaining citizens and the remaining endowment, then the initial prescribed allocation should not change for the remaining citizens. A principle is *consistent* if it satisfies this invariance property. The weaker version of the property obtained by focusing only on subgroups of two citizens is called *bilateral consistency*. A bilateral principle admits a *bilaterally consistent extension* to simple problems if the principle is *bilaterally consistent* and coincides with the bilateral principle for each two-citizen simple problem. Similarly, a bilateral principle admits a *consistent extension* to simple problems if there exists a principle which is *consistent* and coincides with the bilateral principle for each two-citizen simple problem.

We focus on a class of *resource monotonic* and *anonymous* bilateral principles satisfying a regularity condition which was introduced in Dagan and Volij (1997). We build on the extendability of bilateral principles to simple allocation problems (Dagan and Volij 1997) and further extend these bilateral principles to allocation rules on networks. We require the extension to preserve the essence of the bilateral principle for any two-citizen simple allocation problem. We call this condition *pairwise robustness* with respect to the bilateral principle. We provide an algorithm to extend a bilateral principle to an allocation rule which is *pairwise robust*. This algorithm is parallel to the ascending algorithms used in Moulin (1999) and Bochet et al. (2013, 2012). Then, we show that each bilateral principle in this class has a unique extension which is *pairwise robust* with respect to the bilateral principle (Theorem 1).

Next, to demonstrate the global implications of *pairwise robustness*, we define a fairness condition in spirit of the Rawlsian criteria of distributive justice, which states that any inequalities must benefit all citizens, and particularly must benefit those who will receive the least (Rawls 1971). We show that for any consistent extension of a bilateral principle to a simple allocation problem, there is a unique *"Rawls fair"* rule with respect to that consistent extension and this rule is the extension given by the ascending algorithm (Theorem 2). Also, as a corollary, we show that any parametric principle can be extended to an allocation rule (Corollary 1). Finally, we give applications of the algorithm for the egalitarian, the proportional, and the contested garment bilateral principles (Example 1).

The literature on flow sharing on networks has focused on the computation of egalitarian solutions (Megiddo 1974, 1977; Brown 1979; Hall and Vohra 1993). Several allocation rules for allocation problems on networks have recently been introduced and axiomatized in Branzei et al. (2008), Bjørndal and Jörnsten (2010), Bochet et al. (2012, 2013), Moulin and Sethuraman (2013), and Szwagrzak (2011). One way to study allocation rules on networks is to represent the allocation problem as a network flow problem where transfers between nodes are costly and analyze the related minimum cost flow problem on a simple network and implement some known principles for simple allocation problems via suitable cost functions (Branzei et al. 2008). Another way is to look for an extension of bilateral principles for two-person problems to allocation rules (Bjørndal and Jörnsten 2010). Bjørndal and Jörnsten (2010) only focus on the extension of the egalitarian and the contested garment bilateral principles and their network structure is different than ours since they also have links between the sources.

The egalitarian rule for allocation problems on networks has been characterized by *Pareto optimality*, *equal treatment of equals*, and *strategy-proofness* in Bochet et al.

(2013).[4] Their egalitarian rule is an extension of the uniform rule (Sprumont 1991) to allocation problems on networks in an *agent-consistent* way, i.e., if an agent leaves the problem with her assignment and the corresponding amount is reduced from the sources she received her share, then in the remaining network, each remaining agent should receive the same amount as in the original problem.

We want to emphasize that the aim of the paper is not to characterize any allocation rule. We extend a large class of bilateral principles for two-citizen simple allocation problems to allocation rules on a network in an *agent-consistent* manner.[5] Our *pairwise robustness* captures this "consistency" requirement. We implicitly assume that the agents are not held responsible for their connections to the sources. This assumption is reasonable in settings such as blood transfusions, or natural resource networks (where geographical constraints play an important role for the network), or relief aid networks after a natural disaster (where available network after the natural disaster becomes important). An agent should not be responsible for her blood type or for the lack of connections because it might not be possible to construct one or it is not available after a natural disaster.

An alternative extension of simple allocation rules is in a *node-consistent* fashion, i.e., in addition to *agent-consistency*, if a source leaves the problem with its endowment and the corresponding amounts are reduced from the claims of the agents' who were receiving them from the source, then the new problem should assign the agents the amounts they received in the original problem minus the amounts allocated from the deleted source (Moulin and Sethuraman 2013). Another requirement in Moulin and Sethuraman (2013) is *edge-consistency* which requires that if an edge is deleted from the network of the original problem and the agent's claim and the endowment of the source which formed the deleted link is decreased by the amount flowing to the agent from this source, then at the new problem the agent's allocation should be equal to the original allocation minus the amount which the deleted link used to carry. This is a more demanding condition than *node-consistency* and it means that the agents are held responsible for their connections. They show that the proportional principle has a unique extension which satisfies both conditions, the egalitarian principle has infinitely many extensions, and the Talmud principle (Aumann and Maschler 1985) does not have an extension which satisfies both conditions.[6] In general, their alternative notion of extendability does not guarantee existence or uniqueness for those principles which are only weakly resource monotonic, i.e. an increase in total endowment weakly benefits the claimants. Their results motivated our examples in Sect. 4 where we show that all three aforementioned principles have a unique extension which satisfies *pairwise robustness*. Our example also shows that *pairwise robustness* is different from the joint requirement of *node-consistency* and *edge-consistency*. Even though both our paper and Moulin and Sethuraman (2013) give a unique extension of the proportional principle, those extensions are not equivalent.

[4] Szwagrzak (2011) also explores other properties of the egalitarian rule and other rules in this environment.

[5] Note that we use the term "agent" interchangeably with the term "citizen".

[6] We use the definition of the Talmud principle following Aumann and Maschler (1985), which is the *consistent* extension of the contested garment rule.

Bochet et al. (2012) study a similar model where suppliers and demanders of a homogeneous commodity are embedded in a bipartite network. A transfer is possible only between connected agents. Both suppliers and demanders have preferences over the total amount they transfer. That is different from an allocation problem because agents (demanders) receive the commodity from other agents (suppliers).[7] In our model, only one side of the bipartite network is formed by the agents, namely the citizens.

Another related allocation problem is the one in which agents are located sequentially on a line, the so-called river sharing problem (Ambec and Sprumont 2002; Ambec and Ehlers 2008; Ansink and Weikard 2012). A river sharing problem can be written as an allocation problem on a network where agents' access to sources are hierarchical.[8] Hence, our model is more general than a river sharing problem as we have no restrictions on the possible networks between sources and agents. Our model is also different than the division of a single commodity supplied by multiple sources as studied in Kar (2008). In their model, although an agent, a priori, can consume from any source, she must receive all her allocation from a single source whereas an agent in our model can receive her share from several of the sources to which she has access.

In Sect. 2, we introduce the model and some properties of bilateral principles. In Sect. 3, we present the algorithm and give the results. In Sect. 4, we give three applications of the algorithm. We conclude in Sect. 5.

2 Model

For a finite set A, let $|A|$ be its cardinality. Let $\mathcal{S}$ be the finite set of potential sources and $\mathcal{C}$ be the finite set of potential citizens. Let $\mathbf{S} \subseteq \mathcal{S}$ be the set of **sources** and $|S| = m$, and $\mathbf{C} \subseteq \mathcal{C}$ be the set of **citizens** and $|C| = n$. Each **source** $\mathbf{t} \in S$ has a non-negative **endowment** $\mathbf{s_t} \in \mathbb{R}_+$ and each **citizen** $\mathbf{i} \in C$ has a non-negative **claim** $\mathbf{c_i} \in \mathbb{R}_+$ for the resource.

Let $s = (s_1, s_2, \ldots, s_m)$ be the endowment vector and $c = (c_1, c_2, \ldots, c_n)$ be the claims vector. Sources and citizens are embedded in a network in which citizens can acquire the resource only from the sources they are connected to. A **bipartite graph** $\mathbf{g} \subseteq S \times C$ consists of links between **nodes** in S and C. If a **link** in g connects a source t to a citizen i, i.e., $ti \in g$, then it is possible for citizen i to acquire the resource from source t. Let $\mathcal{G}_{m\times n}$ be the set of all connected bipartite graphs between S and C.[9] Let $\mathbf{N_g(T)}$ be the set of citizens connected to a subset of sources $T \subseteq S$

[7] This problem has previously been studied without network constraints in Klaus et al. (1997, 1998) and Kıbrıs and Küçükşenel (2009).

[8] In more detail, the river sharing problem can be written as an allocation problem on a network in the following manner: The initial stream reaching the first agent on the river and the rainfall received by every agent are the sources in our network. The last agent on the river has access to all sources. The second from the last agent has access to all sources except the rainfall of the last agent and in general an agent has access to all sources except the rainfall of her downstream agents.

[9] Throughout the paper, we assume that g is connected. Otherwise, we can treat each connected component of g as a separate problem.

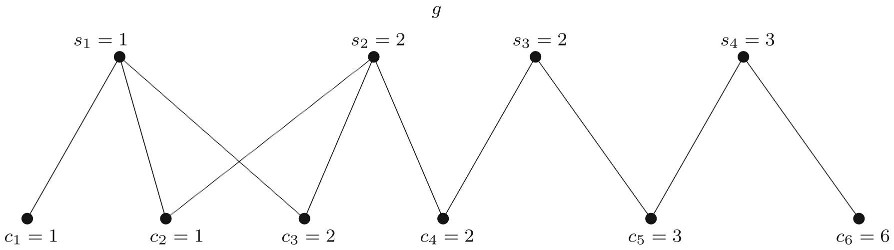

Fig. 1 An example of a problem: $R = (s, c, g)$ is a problem with $S = \{1, 2, 3, 4\}, C = \{1, 2, 3, 4, 5, 6\}$, $s = (1, 2, 2, 3)$, and $c = (1, 1, 2, 2, 3, 6)$

in g, i.e., $N_g(T) = \{i \in C$ such that $ti \in g$ for some $t \in T\}$. Similarly, $\mathbf{N_g(D)}$ be the set of sources connected to a subset of citizens $D \subseteq C$ in g, i.e., $N_g(D) = \{t \in S$ such that $ti \in g$ for some $i \in D\}$.

An allocation problem, simply a **problem**, is a triple $\boldsymbol{R} = (\boldsymbol{s, c, g})$ such that for each $S \subseteq \mathcal{S}$, each $C \subseteq \mathcal{C}$, each $g \in \mathcal{G}_{m\times n}$, and each $T \subset S$, we have $\sum_{t\in T} s_t < \sum_{i\in N_g(T)} c_i$. This means that no subset of sources has enough resource to satisfy the claims of the citizens connected to them. Hence, the problem is "genuine" in the sense that a citizen receives the resource always at the expense of some other citizen. Let $\boldsymbol{\mathcal{R}}(\boldsymbol{S, C}) = \mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathcal{G}_{m\times n}$ be the set of problems for the set of sources S and the set of citizens C. Let $\boldsymbol{\mathcal{R}} = \cup_{S\subseteq\mathcal{S}, C\subseteq\mathcal{C}} \mathcal{R}(S, C)$ be set of all problems. See Fig. 1 for an example.

An **allocation** is a vector $\mathbf{q} = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n_+$ that represents how much resource is allocated to each citizen. A transfer of resources, or simply a **flow**, is a vector $\boldsymbol{\phi} \in \mathbb{R}^{S\times C}_+$, where ϕ_{ti} is the amount sent from source t to citizen i such that if $ti \notin g$, then $\phi_{ti} = 0$.

An allocation q is **feasible** if there is a flow $\phi \in \mathbb{R}^{S\times C}_+$ that supports it, i.e., for each citizen $i \in C$, $q_i = \sum_{t\in N_g(i)} \phi_{ti}$ and for each source $t \in S$, $\sum_{i\in N_g(t)} \phi_{ti} \le s_t$. An allocation q satisfies **claim boundedness** if it is *feasible* and for each citizen $i \in C$, $q_i \le c_i$. An allocation q is **efficient** if it is a *feasible* allocation that satisfies **claim boundedness** and there is no other *feasible* allocation q' such that for each citizen $i \in C$, we have $c_i \ge q'_i \ge q_i$ and $\sum_i q'_i > \sum_i q_i$.

An allocation rule $\boldsymbol{\varphi}$, simply a **rule**, is a function which assigns to each problem $(s, c, g) \in \mathcal{R}$ an *efficient* allocation. Since each rule assigns an allocation to each problem, there is a flow supporting that allocation. If $\phi(s, c, g)$ is a flow that supports $\varphi(s, c, g)$, then for each citizen i, $\varphi_i(s, c, g) = \sum_{t\in N_g(i)} \phi_{ti}(s, c, g)$.

A **simple problem** $\boldsymbol{P} = (\boldsymbol{c}, \boldsymbol{\omega})$ is a problem $(s, c, g) \in \mathcal{R}$ such that $g = S \times C$ and $\sum_{i\in S} s_i = \omega$ with $\sum_{i\in C} c_i > \omega$. Note that P represents the problem of allocating an amount $\omega \ge 0$ among the citizens in C. There is no restriction on the possible flows and ω can be distributed freely. Let $\boldsymbol{\mathcal{P}}(\boldsymbol{S, C}) = \mathbb{R}^n_+ \times \mathbb{R}_+$ be the set of simple problems for the set of sources S and the set of citizens C. Let $\boldsymbol{\mathcal{P}} = \cup_{S\subseteq\mathcal{S}, C\subseteq\mathcal{C}} \mathcal{P}(S, C)$ be the set of all simple problems. A subset $\boldsymbol{\mathcal{P}}_2 = \cup_{S\subseteq\mathcal{S}, C\subseteq\mathcal{C}, |C|=2} \mathcal{P}(S, C)$ of $\mathcal{P}$ is the set of all two-citizen simple problems.

A **bilateral principle** $\boldsymbol{f}$ is a principle that assigns to each two-citizen simple problem an *efficient* allocation, i.e., $\sum_{i\in C} f_i(c, \omega) = \omega$ and for each $i \in C$, $f_i(c, \omega) \le c_i$. A

principle F is a function defined over $\mathcal{P}$ that assigns to each simple problem (c, ω) an *efficient* allocation.

A bilateral principle f is **resource monotonic** if for each pair of two-citizen simple problems $(c, \omega) \in \mathcal{P}_2$ and $(c', \omega') \in \mathcal{P}_2$ with $c = c'$ and $\omega < \omega'$ and each $i \in C$, we have $f_i(c, \omega) \leq f_i(c, \omega')$. *Resource monotonicity* requires that when the resource increases, each citizen receives at least as much as he did initially.

A bilateral principle f is **anonymous** if for each two-citizen simple problem $((c_i, c_j), \omega) \in \mathcal{P}_2$ if $c'_i = c_j$ and $c'_j = c_i$, then $f_i((c_i, c_j), \omega) = f_j((c'_i, c'_j), \omega)$. *Anonymity* requires that the allocation given by the bilateral principle should not depend on the names of the citizens.

Next, we define a property for principles. Suppose a principle has been applied to a simple problem and some citizens leave with what they are prescribed by the principle and the total resource prescribed to these citizens by the principle reduced from the corresponding sources. If we apply the principle to the problem with the remaining citizens and the remaining resources, then the initial prescribed allocation should not change for the remaining citizen. A principle is *consistent* if it satisfies this invariance property. Formally, a principle F is **consistent** if for each $S \subseteq \mathcal{S}$, each $C \subseteq \mathcal{C}$, each simple problem $(c, \omega) \in \mathcal{P}$, each $D \subset C$, and each $i \in C \backslash D$,

$$F_i(c_{-D}, \omega - \sum_{j \in D} F_j(c, \omega)) = F_i(c, \omega)$$

where c_{-D} is the claims vector of the citizens in $C \backslash D$.

The weaker version of the property obtained by focusing only on subgroups of two remaining citizens is called **bilateral consistency**. A bilateral principle f admits a **bilaterally consistent extension** F to simple problems if F is *bilaterally consistent* and coincides with f for each two-citizen simple problem. Similarly, a bilateral principle f admits a **consistent extension** F to simple problems if F is *consistent* and coincides with f for each two-citizen simple problem.

Let f be a bilateral principle, $P = (c, \omega)$ be a simple problem, and q be an allocation satisfying claim-boundedness. The binary relation $\succ_{f,q}$ over the set of citizens C is defined as in Dagan and Volij (1997):

$$\succ_{f,q} = \{(i, j) \in C \times C | f_i((c_i, c_j), q_i + q_j) < q_i\}.$$

A bilateral principle f is **regular**, if for each simple problem P and each allocation q satisfying *claim boundedness*, the binary relation $\succ_{f,q}$ is *transitive*.

Dagan and Volij (1997) offer *regularity* as a necessary and sufficient condition on a bilateral principle f which guarantees the existence of a *consistent extension* of f to simple allocation problems. The binary relation reveals which of the two citizens are treated more favorably in a simple allocation problem in comparison to how the bilateral principle treats them in a two-citizen simple allocation problem. The *regularity* condition requires this comparison to be transitive. In their Theorem 3.2, Dagan and Volij (1997) show that a resource monotonic and anonymous bilateral principle has a unique extension to simple allocation problems if and only if it satisfies *regularity*. That extension delivers the allocation in which the binary relation gives an equivalence between any pair of citizens.[10] We build on this result to extend bilateral principles further to allocation problems where the distribution is subject to network constraints.

We focus on bilateral principles that satisfy *resource monotonicity*, *anonymity*, and *regularity*. Our aim is to extend such a bilateral principle to a rule on a network. We require the extension to preserve the essence of the bilateral principle. We formalize this requirement by the following definition[11]:

Pairwise f-Robustness: Let $R = (s, c, g) \in \mathcal{R}$ be a problem and f be a bilateral principle. A *feasible* allocation q is *pairwise f-robust* if for each pair of citizens $i, j \in C$ with $f((c_i, c_j), q_i + q_j) = (q_i^*, q_j^*)$, there exists no *feasible* allocation q' for R such that for each $k \neq i, j$, $q'_k = q_k$ and

$$|q_i^* - q_i'| < |q_i^* - q_i| \text{ or } |q_j^* - q_j'| < |q_j^* - q_j|.$$

A rule φ is **pairwise *f*-robust** if for each problem $(s, c, g) \in \mathcal{R}$, $\varphi(s, c, g) = q$ satisfies *pairwise f-robustness*.

The *pairwise robustness* condition is only concerned with the allocations of pairs of citizens. To demonstrate the global implications of *pairwise robustness*, we define a fairness condition on the overall allocation in spirit of the Rawlsian criteria of distributive justice. To do so, first we need to introduce some notation. For each $x \in \mathbb{R}^n$, let x^* be the *order statistics* of x, obtained by rearranging the coordinates of x in increasing order: $x_1^* \leq x_2^* \leq \cdots \leq x_n^*$. An allocation **x Rawls dominates y** if there exists $k \in \{1, \ldots, n\}$ such that $x_k^* \geq y_k^*$ and for each l with $1 \leq l < k$, $x_l^* = y_l^*$. We denote this as **x RD y**.

Let $R = (s, c, g)$ be a problem, q be an allocation, F be a principle, and i be a citizen. The *welfare observed by a citizen i at R from q with respect to F* is $w_i(c, q, F) = \sup\{\omega \,|\, q_i = F(c, \omega)\}$. At an allocation q, the citizen gets exactly what she would have gotten at a simple problem $(c, w_i(c, q, F))$ under principle F. Since there are network constraints at R which hinder the free flow of resources, we use $w_i(c, q, F)$ as a measure of how well the citizen i is treated at allocation q when principle F is used as the allocation criteria. Let $w(c, q, F)$ be the vector of welfare obtained by the agents in C. Let $w^*(c, q, F)$ be the order statistics of $w(c, q, F)$.

Rawls F-Fairness: Let $R = (s, c, g)$ be a problem and F be a principle. A *feasible* allocation q is *Rawls F-fair* if for each *feasible* allocation $q' \neq q$,

$$w^*(c, q, F) \; RD \; w^*(c, q', F).$$

A rule φ is **Rawls *F*-fair** if for each problem $(s, c, g) \in \mathcal{R}$, $\varphi(s, c, g) = q$ is *Rawls F-fair*.

[10] We refer the reader to Dagan and Volij (1997) for an in-depth analysis of this condition and its implications.

[11] Since each bilateral principle assigns an efficient allocation by definition, this condition is equivalent to

$$|q_i^* - q_i'| < |q_i^* - q_i| \text{ and } |q_j^* - q_j'| < |q_j^* - q_j|.$$

3 The ascending algorithm

Next, we construct an algorithm to extend each *resource monotonic, anonymous*, and *regular* bilateral principle f to a rule φ^f that is *pairwise* f*-robust*. We also show that for each bilateral principle f, there is a unique *pairwise* f*-robust* rule. If f is a bilateral principle that satisfies *resource monotonicity, anonymity*, and *regularity*, then f has a unique *bilaterally consistent extension* F to simple problems (Aumann and Maschler 1985; Dagan and Volij 1997). If a *resource monotonic, anonymous*, and *regular* bilateral principle f admits a *bilaterally consistent extension* F, then F also satisfies *resource monotonicity* (Hokari and Thomson 2008). Also, F is *consistent* (Dagan and Volij 1997; Chun 1999).

Given the claims of the citizens c, consider the simple problem with a single source of capacity $\omega \geq 0$. For each citizen $i \in C$, let $F_i(c, \omega)$ be the amount that citizen i would have received under the principle F in the simple problem (c, ω). Note that by *resource monotonicity*, $F_i(c, \omega)$ is increasing in ω.

We obtain the rule φ^f by an ascending algorithm based on the following system $K(\omega)$ of inequalities where ω is a non-negative parameter:

$$\sum_{i \in D} F_i(c, \omega) \leq \sum_{t \in N_g(D)} s_t \text{ for each } D \subseteq C. \tag{1}$$

For $\omega = 0$, Eq. (1) is satisfied for each $D \subseteq C$. For $\omega = \sum_{i \in C} c_i$, there exists $D \subseteq C$ such that

$$\sum_{i \in D} F_i(c, \omega) > \sum_{t \in N_g(D)} s_t$$

by construction. Hence, there exists a largest ω^1 such that

$$\sum_{i \in D} F_i(c, \omega^1) \leq \sum_{t \in N_g(D)} s_t \text{ for each } D \subseteq C \text{ and} \tag{2}$$

$$\sum_{i \in D} F_i(c, \omega^1) = \sum_{t \in N_g(D)} s_t \text{ for some } D \subseteq C. \tag{3}$$

Since $\sum_{t \in N_g(D)} s_t$ is a submodular function of D, there exists a unique largest D^1 such that Eq. (3) holds. The allocation for the agents in D^1 is obtained by setting

$$q_i = F_i(c, \omega^1) \text{ for each } i \in D^1.$$

If $D^1 = C$, the algorithm stops. Otherwise, the algorithm continues to assign allocations to the agents in the rest of the network $(S \setminus N_g(D^1), C \setminus D^1, g \setminus (N_g(D^1) \times D^1))$. That is, we look for the largest $\omega^2 > 0$ such that

$$\sum_{i \in D} F_i(c, \omega^2) \leq \sum_{t \in N_g(D) \setminus N_g(D^1)} s_t \text{ for each } D \subseteq C \setminus D^1 \text{ and} \tag{4}$$

$$\sum_{i \in D} F_i(c, \omega^2) = \sum_{t \in N_g(D) \setminus N_g(D^1)} s_t \quad \text{for some } D \subseteq C \setminus D^1. \tag{5}$$

Then, there exists a unique largest set D^2 such that Eq. (5) holds. Observe that $\omega^2 > \omega^1$. Because if $\omega^2 \leq \omega^1$, we can combine Eqs. (3) and (5) to obtain

$$\sum_{i \in D^1 \cup D^2} q_i(c, \omega^1) \geq \sum_{i \in D^1} q_i(c, \omega^1) + \sum_{i \in D^2} q_i(c, \omega^2) = \sum_{t \in N_g(D^1 \cup D^2)} s_t$$

which contradicts the choice of D^1 as the largest set satisfying Eq. (3).

If $D^1 \cup D^2 = C$, then the algorithm stops. Otherwise, the algorithm continues iteratively to assign all agents their allocations. Since the network is finite, the algorithm stops after a finite number of iterations.

The rule φ^f assigns the allocation q obtained in the algorithm above.

Theorem 1 *For each resource monotonic, anonymous, and regular bilateral principle f, φ^f extends f to a pairwise f-robust rule . Moreover, the extension is unique.*

We prove Theorem 1 in three lemmas:

Lemma 1 (Theorem 3.2 in Dagan and Volij 1997) *A resource monotonic, anonymous, and regular bilateral principle f has a unique consistent extension F to simple problems.*

Lemma 2 *Suppose that a resource monotonic bilateral principle f has a consistent extension F to simple problems. Then, φ^f extends f to a pairwise f-robust rule.*

Proof First, we show that the rule φ^f obtained by the ascending algorithm is *pairwise f-robust*. Let (s, c, g) be a problem, $\varphi^f(s, c, g) = q$, and $i, j \in C$. First, suppose that the ascending algorithm assigns to i and j their allocations in the same iteration step, i.e., for the parameter obtained from the ascending algorithm, $\omega > 0$, we have $q_i = F_i(c, \omega)$ and $q_j = F_j(c, \omega)$. Consider the simple problem $((c_i, c_j), q_i + q_j)$. Since F is *consistent*, $F_{\{i,j\}}(c, \omega) = f((c_i, c_j), q_i + q_j) = (q_i, q_j)$. Hence, *pairwise f-robustness* is trivially satisfied.

Next, suppose that the ascending algorithm assigns to i and j their allocations in different iteration steps, i.e., for two different parameters obtained from the ascending algorithm, $\omega, \tilde{\omega} > 0$, we have $q_i = F_i(c, \omega)$ and $q_j = F_j(c, \tilde{\omega})$. Assume without loss of generality that $\tilde{\omega} > \omega$.

If a *resource monotone* bilateral principle f has a *consistent extension* F, then F also satisfies *resource monotonicity* (Hokari and Thomson 2008). By *resource monotonicity*, we have $q_i = F_i(c, \omega) \leq F_i(c, \tilde{\omega})$ and $q_j = F_j(c, \tilde{\omega}) \geq F_j(c, \omega)$. If one of these inequalities is not strict, then $F_{\{i,j\}}(c, \omega) = f((c_i, c_j), q_i + q_j) = (q_i, q_j)$ by the same argument presented above. Hence, assume that $q_i = F_i(c, \omega) < F_i(c, \tilde{\omega})$ and $q_j = F_j(c, \tilde{\omega}) > F_j(c, \omega)$. Let $F_i(c, \tilde{\omega}) = \bar{q}_i$ and $F_j(c, \omega) = \bar{q}_j$. By *consistency*, $F_{\{i,j\}}(c, \omega) = f((c_i, c_j), q_i + \bar{q}_j) = (q_i, \bar{q}_j)$ and $F_{\{i,j\}}(c, \tilde{\omega}) = f((c_i, c_j), \bar{q}_i + q_j) = (\bar{q}_i, q_j)$. Then, by *resource monotonicity*, $f_i((c_i, c_j), q_i + q_j) \geq q_i$ and $f_j((c_i, c_j), q_i + q_j) \leq q_j$.

Next, we show that it is not possible to increase i's allocation when the allocations of all citizens other than i and j remain the same. The algorithm assigns j's allocation at some step $\tilde{h}$ after assigning i's allocation at some step h, i.e. $\tilde{h} > h$. The citizens who receive their allocations at step h obtain no resource from the sources they share with citizens who receive their allocations earlier and receive all the resources of their other connections (as given in Eqs. 3 and 5). The citizens who receive their allocations at step h or earlier deplete all the sources they have access to. This implies that if q' is a feasible allocation which coincides with q for citizens different from i and j, then necessarily $q'_i \leq q_i$. Hence, φ^f obtained by the ascending algorithm is *pairwise f-robust.* □

Lemma 3 *Suppose that a resource monotonic bilateral principle f has a consistent extension F to simple problems. Then, φ^f is the unique extension of f that is pairwise f-robust.*

Proof Let φ^f be the rule given by the ascending algorithm. Suppose there exists some other rule $\varphi \neq \varphi^f$ which is also *pairwise f-robust*. Then, there exists a problem (s, c, g) such that $\varphi(s, c, g) = q' \neq q = \varphi^f(s, c, g)$.

Let $D^1, D^2, \ldots, D^h$ be the sets of citizens which are allocated in steps $1, 2, \ldots, h$ of the ascending algorithm, respectively and let $\omega^1 < \omega^2 < \ldots < \omega^h$ be the corresponding parameters associated to the sets $D^1, D^2, \ldots, D^h$ in the ascending algorithm. Suppose that there exists $i \in D^1$ such that $q'_i < q_i$. By construction, $q_i = F_i(c, \omega^1)$.

Let ϕ' be a flow which supports the allocation q'. Consider the set of sources $S^1 = N_g(i)$ connected to i. Let $C^1 = \{j \in C : \exists t \in S^1 \text{ such that } \phi'_{tj} > 0\}$. This set is non-empty since the resources which were assigned to i in q must now be allocated to other citizens who share sources with i and the allocation q' is *efficient*.

Suppose that there exists $j \in C^1$ such that $q'_j > q_j$. The citizen j receives her allocation at some step d of the algorithm, for $1 \leq d \leq h$ with the associated parameter ω^d. Since $d \geq 1, \omega^d \geq \omega^1$ and $q'_j > q_j = F_j(c, \omega^d) \geq F_j(c, \omega^1)$ by resource monotonicity of extension F. Let $\bar{q}_i = F_i(c, \omega^d)$. Since $d \geq 1, \bar{q}_i \geq q_i > q'_i$. Due to the consistency of F, $f_j((c_i, c_j), \bar{q}_i + q_j) = q_j$ and $f_i((c_i, c_j), \bar{q}_i + q_j) = \bar{q}_i$. If $q'_i + q'_j \geq \bar{q}_i + q_j$, then by the resource monotonicity of f, $f_i((c_i, c_j), q'_i + q'_j) \geq \bar{q}_i > q'_i$ and $f_j((c_i, c_j), q'_i + q'_j) < q'_j$. If, on the contrary, $q'_i + q'_j < \bar{q}_i + q_j$, then, again, by the resource monotonicity of f, $f_j((c_i, c_j), q'_i + q'_j) \leq q_j < q'_j$ and $f_i((c_i, c_j), q'_i + q'_j) > q'_i$.

It is possible to change allocation q' by transferring some arbitrarily small ε amount from j to i through the path jt, ti without changing the allocations of citizens other than i and j. Since the link jt carries a positive flow in ϕ', such a transfer is possible, contradicting *pairwise f-robustness*.

Next, suppose that for each citizen $j \in C^1, q'_j \leq q_j$. Consider the set $S^2 = N_g(C^1)$ and $C^2 = \{j \in C : \exists t \in S^2 \text{ such that } \phi'_{tj} > 0\}$. Since $q'_i < q_i$ and for each citizen $j \in C^1, q'_j \leq q_j, C^2$ is non-empty because the resources received by i and agents in

C^1 at q must now be allocated to other citizens who share sources with those agents, due to the *efficiency* of the allocation q'.

Suppose that there exists $j_2 \in C^2$ such that $q'_{j_2} > q_{j_2}$. By the same argument above, we can conclude $f_{j_2}((c_i, c_{j_2}), q'_i + q'_{j_2}) < q'_{j_2}$ and $f_i((c_i, c_{j_2}), q'_i + q'_{j_2}) > q'_i$. It is possible to change allocation q' by transferring some arbitrarily small ε amount from j_2 to i through a path $j_2t_2, t_2j_1, j_1t_1, t_1i$ for some $j_2 \in C^2, t_2 \in S^2, j_1 \in C^1$, and $t_1 \in S^1$, without changing the allocations of citizens other than i and j_2, contradicting *pairwise f-robustness*.

If there exists no $j \in C^2$ such that $q'_j > q_j$, then we continue iteratively to look for a j such that $q'_j > q_j$. Such a j exists since $q'_i < q_i$ and the rule φ is *efficient*. Again, we can conclude by resource monotonicity $f_j((c_i, c_j), q'_i + q'_j) < q'_j$ and $f_i((c_i, c_j), q'_i + q'_j) > q'_i$. Using the construction in the paragraph above, it is possible to make a transfer from j to i without changing the allocations of other citizens through path, contradicting *pairwise f-robustness*.

If for each $i \in D^1, q'_i \geq q_i$, then we have $q'_i = q_i$ for all $i \in D^1$. Since $q \neq q'$, there exists $i \in D^{h_1}$ such that for each $h_2 < h_1$ and each $j \in D^{h_2}, q'_j = q_j$ and we can apply the same iterative argument starting from i to find a contradiction to *pairwise f-robustness*.

Hence, φ^f is the unique rule which is *pairwise f-robust*. □

Next, we consider Rawls' criteria of justice and show that there is a unique *Rawls F-fair* rule that is equal to the extension given by the algorithm.

Theorem 2 *Let f be a resource monotonic, anonymous, and regular bilateral principle and let F be the unique consistent extension of f to simple problems. Then, φ^f is the unique Rawls F-fair rule.*

Proof Let f be a resource monotonic, anonymous, and regular bilateral principle and let F be the unique consistent extension of f to simple problems. Let φ^f be the rule given by the ascending algorithm. Next, we show that φ^f is the unique Rawls F-fair rule.

Let $R = (s, c, g)$ be a problem and $\varphi^f(s, c, g) = q$. Let q' be a feasible allocation such that $q' \neq q$. Let $D^1, D^2, \ldots, D^h$ be the sets of citizens which are allocated in steps $1, 2, \ldots, h$ of the ascending algorithm, respectively and let $\omega^1 < \omega^2 < \cdots < \omega^h$ be the corresponding parameters associated to the sets $D^1, D^2, \ldots, D^h$ in the ascending algorithm. Since $q \neq q'$, there is a citizen $i \in D^k$ for some $1 \leq k \leq h$ such that $q_i > q'_i$ and for each $1 \leq l < k$, each citizen $j \in D^l, q_j = q'_j$. Such a citizen $i \in D^k$, which receives less in q' than in q, exists because at any flow which supports q the citizens in D^k do not receive any resource from the sources they share with a citizen $j \in D^l$ for $1 \leq l < k$ and exclusively deplete any source they do not share with a citizen $j \in D^l$ for $1 \leq l < k$.

By *resource monotonicity*, for some $0 \leq \omega' < \omega^k$, we have $q_i = F_i(c, \omega^k)$ and $q'_i = F_i(c, \omega')$. Hence, $w^*(c, q, F)\ RD\ w^*(c, q', F)$. Therefore, φ^f is the unique *Rawls F-fair* rule. □

Finally, we define a family of principles. Before defining the family, we introduce a fairness property. A principle satisfies *equal treatment of equals* if two citizens with

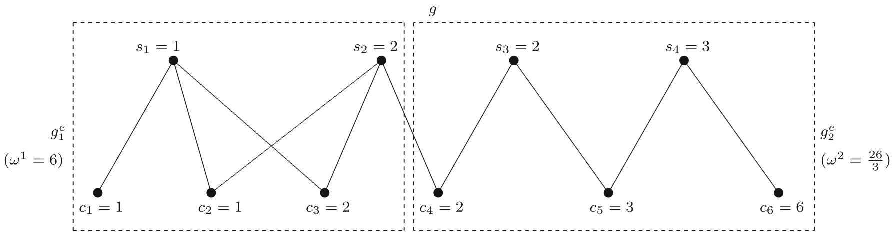

Fig. 2 Egalitarian rule

equal claims receive equal amounts. Formally, F satisfies **equal treatment of equals** if for each simple problem $(c, \omega) \in \mathcal{P}$ and each pair of citizens $i, j \in C$ with $c_i = c_j$, we have $F_i(c, \omega) = F_j(c, \omega)$.

Let Φ be the family of functions $\phi : \mathbb{R}_+ \times [\underline{\lambda}, \overline{\lambda}] \to \mathbb{R}_+$, where $-\infty \leq \underline{\lambda} \leq \overline{\lambda} \leq \infty$, that are continuous, nowhere decreasing with respect to the second argument, and such that for each $c_0 \in \mathbb{R}_+$, we have $\phi(c_0, \underline{\lambda}) = 0$ and $\phi(c_0, \overline{\lambda}) = c_0$.

Parametric principle of parametrization $\boldsymbol{\phi \in \Phi}$, $\boldsymbol{F^\phi}$ Let $\phi : \mathbb{R}_+ \times [\underline{\lambda}, \overline{\lambda}] \to \mathbb{R}_+ \in \Phi$ is given. Then, for each $(c, \omega) \in \mathcal{P}$, F^ϕ selects the vector $x \in \mathbb{R}^n_+$ such that $\sum_{i \in C} x_i = \omega$ and there is $\lambda \in [\underline{\lambda}, \overline{\lambda}]$ such that for each $i \in C$, $x_i = \phi(c_i, \lambda)$.

These principles are introduced and characterized by Young (1987). A principle is a parametric principle if and only if it satisfies *continuity, equal treatment of equals*, and *bilateral consistency*. We also know that any parametric principle is *resource monotonic* (Young 1987). Hence, by rearranging the proof of Theorem 1, we have the following corollary:

Corollary 1 *Any parametric principle f for simple problems is uniquely extendable to a pairwise f-robust rule.*

4 An example with three rules

We apply the ascending algorithm to the example in Fig. 1 to extend the egalitarian, the proportional, and the contested garment bilateral principles. The examples help to illustrate that while extending different bilateral principles, the ascending algorithm might result in different decompositions and even when the decompositions are the same, the final allocations are different.

Example 1 Consider the problem $R = (s, c, g)$ with $S = \{1, 2, 3, 4\}, C = \{1, 2, 3, 4, 5, 6\}$, $s = (1, 2, 2, 3)$, $c = (1, 1, 2, 2, 3, 6)$, and g is given in Fig. 1.

The first rule is the egalitarian rule based on egalitarian bilateral principle.

Egalitarian Bilateral Principle, $\boldsymbol{e}$: For each $P = (c, \omega) \in \mathcal{P}_2$, the egalitarian bilateral principle assigns the allocation $\boldsymbol{e}(\boldsymbol{c}, \boldsymbol{\omega}) = q$ such that for each $i \in C$, $q_i = min\{c_i, \lambda\}$ where λ solves $\sum_{i \in C} min\{c_i, \lambda\} = \omega$.

The ascending algorithm which extends the egalitarian bilateral principle decomposes the graph into two parts after two iterations. After the first iteration, $\omega^1 = 6$ and $D^1 = \{1, 2, 3\}$. The ascending algorithm terminates after the second itera-

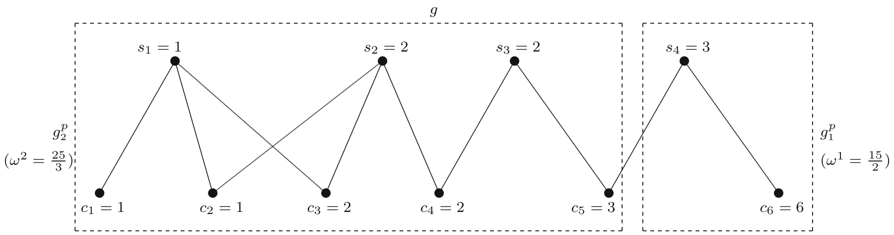

Fig. 3 Proportional rule

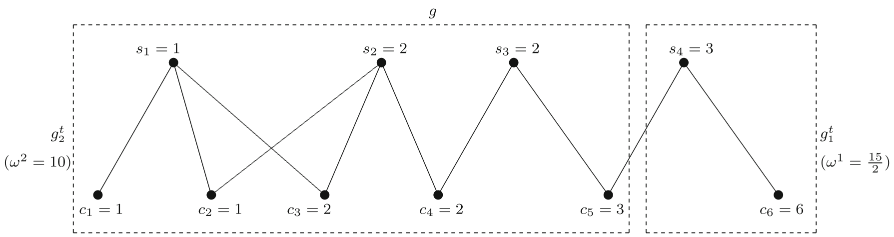

Fig. 4 Talmud rule

tion with $\omega^2 = \frac{26}{3}$ and $D^2 = \{4, 5, 6\}$. The egalitarian rule assigns $\varphi^e(s, c, g) = \left(1, 1, 1, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right)$ (Fig. 2).

The second rule is the proportional rule based on proportional bilateral principle.

Proportional Bilateral Principle, *p*: For each $P = (c, \omega) \in \mathcal{P}_2$, the proportional bilateral principle assigns the allocation $\boldsymbol{p}(\boldsymbol{c}, \boldsymbol{\omega}) = q = \pi c$ where $\pi = \frac{\omega}{\sum_{i \in C} c_i}$.

The ascending algorithm which extends the proportional bilateral principle decomposes the graph into two parts after two iterations . After the first iteration, $\omega^1 = \frac{15}{2}$ and $D^1 = \{6\}$. The ascending algorithm terminates after the second iteration with $\omega^2 = \frac{25}{3}$ and $D^2 = \{1, 2, 3, 4, 5\}$. The proportional rule assigns $\varphi^p(s, c, g) = \left(\frac{5}{9}, \frac{5}{9}, \frac{10}{9}, \frac{10}{9}, \frac{15}{9}, 3\right)$ (Fig. 3).

The third rule is the Talmud rule based on the contested garment bilateral principle. The *consistent* extension of the contested garment bilateral principle is the Talmud principle (Aumann and Maschler 1985). To define this bilateral principle, we first define the equal-sacrifice bilateral principle. For each $P = (c, \omega) \in \mathcal{P}_2$, the equal-sacrifice bilateral principle assigns the allocation $\boldsymbol{l}(\boldsymbol{c}, \boldsymbol{\omega}) = q$ such that for each $i \in C$, $q_i = max\{0, c_i - \sigma\}$ where σ solves $\sum_{i \in C} max\{0, c_i - \sigma\} = \omega$.

Contested Garment Bilateral Principle, *t*: For each $P = (c, \omega) \in \mathcal{P}_2$, the contested garment bilateral principle assigns the allocation $\boldsymbol{t}(\boldsymbol{c}, \boldsymbol{\omega}) = q$ such that for each $i \in C$, $q_i = e\left(\frac{c}{2}, min\{\frac{c_i + c_j}{2}, \omega\}\right) + l\left(\frac{c}{2}, max\{0, \omega - \frac{c_i + c_j}{2}\}\right)$.

The ascending algorithm which extends the contested garment bilateral principle decomposes the graph into two parts after two iterations as seen in Fig. 3. After the first iteration, $\omega^1 = \frac{15}{2}$ and $D^1 = \{6\}$. At the next iteration, $\omega^2 = 10$ and $D^2 = \{1, 2, 3, 4, 5\}$. The Talmud rule assigns $\varphi^t(s, c, g) = \left(\frac{1}{2}, \frac{1}{2}, 1, 1, 2, 3\right)$ (Fig. 4).

5 Conclusion

Our results expand the scope of the existing literature on claims problems. We provide a way to apply bilateral principles to problems on networks of sources and citizens. The extension exists and is unique for a large class of bilateral principles. Such problems with multiple sources are very commonly observed as exemplified in the introduction. Our extension satisfies two equivalent fairness conditions. The *pairwise robustness* is concerned with the allocation of any pair of players while *Rawls fairness* is a condition on the whole allocation. We hope that these two conditions help to understand one another and how our extension functions on both a local and a global scale.

Moreover, the network model raises new theoretical problems. All the questions one can ask for simple allocation problems are relevant for the network setting and there are new issues due to the richness of the structure. Our extension shows that most widely used principles of allocation are relevant also for this setting. Bochet et al. (2012, 2013) and Szwagrzak (2011) have successfully adopted some axioms originally defined for simple allocation problems (e.g. equal treatment of equals, replacement principle etc.) and characterize some allocation rules for network allocation problems. But many more questions remain than those already answered. We would like to underline two such questions:

- Is it possible to characterize rules on the basis of how they respond to changes in the network structure?
- The dual of a problem (Thomson 2006) is well-defined when there is only one source. Can the dual of a network allocation problem be defined?

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